1) *Quickies!* You don't need to justify your answers.

(a) If A is square matrix and $\det A = 0$, what can you say about the number of solutions of $A\mathbf{x} = \mathbf{b}$?

Answer. It is either infinite or there are no solutions. □

(b) If A is square matrix and for some vector $\mathbf{b}$ the system $A\mathbf{x} = \mathbf{b}$ has no solution, then what can you say about the number of solutions of $A\mathbf{x} = \mathbf{0}$? [Here, $\mathbf{0}$ is the zero vector.]

Answer. It is infinite. □

(c) If $A = \begin{bmatrix} 2 & \sqrt{3} & -\pi & 12/131 \\ 0 & -1 & e^2 & \ln(10) \\ 0 & 0 & 7 & \cos(3) \\ 0 & 0 & 0 & 1 \end{bmatrix}$, then what are the $(4,1)$ and $(3,3)$ entries of A^{-1}?

Answer. They are 0, as A^{-1} is upper triangular, and $1/7$ respectively. □

(d) If $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions, then what can be said [for sure] about the reduced [row] echelon form of A?

Answer. There is at least one column without a leading one. □

(e) Write $\begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ as a product of elementary matrices.

Answer.
$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$
□

2) Solve the systems $A\mathbf{x} = \mathbf{b}$ below. (These should be quick and you do not have to show work.)

(a) $A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 3 \\ 0 \end{bmatrix}$.

Answer. $x_4 = 3$, $x_3 = 0$, $x_2 = -1 - x_4 = -4$, $x_1 = 1$. □

(b) $A = \begin{bmatrix} 2 & 3 & -1 & 4 & 5 \\ 0 & 0 & 2 & -1 & 3 \\ 1 & 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 3 \end{bmatrix}$.

Answer. No solution, as the bottom row gives $0 = 3$. □

(c) $A = \begin{bmatrix} 1 & 0 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 2 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}$.

Answer. $x_7 = 1$, $x_6 = t$, $x_5 = -1 - 2t$, $x_4 = s$, $x_3 = t$, $x_2 = -s$, $x_1 = 2 - 2s - t$, where $t, s \in \mathbb{R}$. □

3) Find the inverse of $A = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 3 & 1 & 5 & 6 \\ 0 & 2 & -1 & 8 \\ 2 & 0 & 4 & 3 \end{bmatrix}$.

Solution. We have:

$$\left[\begin{array}{rrrr|rrrr} 1 & 0 & 2 & 1 & 1 & 0 & 0 & 0 \\ 3 & 1 & 5 & 6 & 0 & 1 & 0 & 0 \\ 0 & 2 & -1 & 8 & 0 & 0 & 1 & 0 \\ 2 & 0 & 4 & 3 & 0 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{rrrr|rrrr} 1 & 0 & 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 3 & -3 & 1 & 0 & 0 \\ 0 & 2 & -1 & 8 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{rrrr|rrrr} 1 & 0 & 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 3 & -3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 6 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{rrrr|rrrr} 1 & 0 & 0 & 0 & -17 & 4 & -2 & 3 \\ 0 & 1 & 0 & 0 & 13 & -1 & 1 & 5 \\ 0 & 0 & 1 & 0 & 10 & -2 & 1 & -2 \\ 0 & 0 & 0 & 1 & -2 & 0 & 0 & 1 \end{array}\right].$$

Hence,

$$A^{-1} = \begin{bmatrix} -17 & 4 & -2 & 3 \\ 13 & -1 & 1 & -5 \\ 10 & -2 & 1 & -2 \\ -2 & 0 & 0 & 1 \end{bmatrix}.$$

$\square$

4) Let $A = \begin{bmatrix} 0 & -1 \\ 2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}$, and $C = \begin{bmatrix} 1 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix}$. Compute $((A+2B)\cdot C)^{\mathrm{T}}$.

Solution. We have

$$2B = \begin{bmatrix} 2 & 2 \\ 2 & -4 \end{bmatrix}, \qquad A+2B = \begin{bmatrix} 2 & 1 \\ 4 & -1 \end{bmatrix}, \qquad (A+2B)\cdot C = \begin{bmatrix} 3 & 7 & 3 \\ 3 & 11 & 9 \end{bmatrix}.$$

So,

$$((A+2B)\cdot C)^{\mathrm{T}} = \begin{bmatrix} 3 & 3 \\ 7 & 11 \\ 3 & 9 \end{bmatrix}$$

□

5) Let $A = \begin{bmatrix} 0 & 1 & 7 & 1 & 0 \\ 2 & 5 & -1 & 3 & 0 \\ -1 & 2 & 1 & 5 & 1 \\ 3 & 1 & 3 & -1 & 0 \\ -1 & 1 & -3 & 2 & 0 \end{bmatrix}$.

(a) What is the cofactor of A at position $(3,3)$?

Solution. It is:

$$(-1)^{3+3} = \begin{vmatrix} 0 & 1 & 1 & 0 \\ 2 & 5 & 3 & 0 \\ 3 & 1 & -1 & 0 \\ -1 & 1 & 2 & 0 \end{vmatrix} = 0.$$

[To compute the determinant, use the last column.]

□

(b) If $B = [b_{i,j}]$ is the adjoint of A, then what is $b_{3,2}$? [Be careful here!]

Solution. The entry of the adjoint at $(3,2)$ is the cofactor at $(2,3)$. So, it is

$$(-1)^{2+3} \begin{vmatrix} 0 & 1 & 1 & 0 \\ -1 & 2 & 5 & 1 \\ 3 & 1 & -1 & 0 \\ -1 & 1 & 2 & 0 \end{vmatrix} = -1 \left((-1)^{2+4} \begin{vmatrix} 0 & 1 & 1 \\ 3 & 1 & -1 \\ -1 & 1 & 2 \end{vmatrix} \right) = -1(4-5) = 1.$$

[The first determinant was computed by using the last column.]

□