Dual Basis for the Trace

Math 652

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Let L/K be a finite separable extension and $\mathcal{B} = \{x_1, \dots, x_n\}$ be a K-basis of L. Let $M_{\mathcal{B}} \stackrel{\text{def}}{=} [\mathrm{t}_{L/K}(x_i\, x_j)]$ [with $1 \leq i, j \leq n$]. Then, since $\mathrm{t}_{L/K}$ is K-linear, if $x = \sum_i a_i x_i$ and $y = \sum_i b_i x_i$, then

$$\mathrm{t}_{L/K}(xy) = \sum_i a_i\, \mathrm{t}_{L/K}(x_i y) = \sum_i a_i \left[\sum_j b_j\, \mathrm{t}_{L/K}(x_i x_j) \right].$$

Note that

$$[\mathrm{t}_{L/K}(x_i x_j)] \cdot \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} \sum_j b_j\, \mathrm{t}_{L/K}(x_1 x_j) \\ \vdots \\ \sum_j b_j\, \mathrm{t}_{L/K}(x_n x_j) \end{bmatrix}$$

and hence

$$\mathrm{t}_{L/K}(xy) = \begin{bmatrix} a_1 & \cdots & a_n \end{bmatrix} \cdot [\mathrm{t}_{L/K}(x_i x_j)] \cdot \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}.$$

Note that $[a_1, \dots, a_n]$ is just the coordinates of x with respect to the basis $\mathcal{B}$, usually denoted by $[x]_{\mathcal{B}}$, and that $[b_1, \dots, b_n] = [y]_{\mathcal{B}}$. So, with a choice of $\mathcal{B}$ as above, we have that

$$\mathrm{t}_{L/K}(xy) = [x]_{\mathcal{B}} \cdot M_{\mathcal{B}} \cdot [y]_{\mathcal{B}}^{\mathrm{T}}.$$

Proposition. *There is* $\{y_1, \dots, y_n\} \subseteq L$ *such that* $\mathrm{t}_{L/K}(x_i y_j) = \delta_{ij}$.

Proof. We've seen [see (1.25.b) on pg. 15] that $\det M_{\mathcal{B}} \neq 0$, i.e., $M_{\mathcal{B}}$ is invertible. So, if

$$\vec{e}_i \stackrel{\text{def}}{=} [0, \dots, 1, \dots, 0], \quad 1 \text{ in the } i\text{-th coordinate},$$

let

$$\begin{bmatrix} c_{j,1} \\ \vdots \\ c_{j,n} \end{bmatrix} \stackrel{\text{def}}{=} M_{\mathcal{B}}^{-1} \cdot (\vec{e}_j)^{\mathrm{T}}$$

and $y_j \stackrel{\text{def}}{=} \sum_k c_{j,k} x_k$. Thus, we have $[y_j]_{\mathcal{B}}^{\mathrm{T}} = [c_{j,1}, \ldots, c_{j,n}]^{\mathrm{T}} = M_{\mathcal{B}}^{-1}(\vec{e}_j)^{\mathrm{T}}$.
Then,

$$\mathrm{t}_{L/K}(x_i y_j) = [x_i]_{\mathcal{B}} \cdot M_{\mathcal{B}} \cdot [y_j]_{\mathcal{B}}^{\mathrm{T}} = \vec{e}_i \cdot M_{\mathcal{B}} \cdot (M_{\mathcal{B}}^{-1} \cdot (\vec{e}_j)^{\mathrm{T}}) = \vec{e}_i \cdot (\vec{e}_j)^{\mathrm{T}} = \delta_{ij}.$$

$\square$