Final (In Class Part)

M552 – Abstract Algebra

May 16th, 2008

1. Let R be the ring of real continuous functions $f(x)$ such that $f(x+\pi) = f(x)$, and M be the R-module of real continuous functions $g(x)$ such that $g(x+\pi) = -g(x)$. Let c and s be the usual cosine and sine functions [in M].

(a) Show that $R \not\cong M$ [as R-modules]. [**Hint:** Use calculus.]

(b) Show that $(f, g) \mapsto (fc + gs, -fs + gc)$ is an isomorphism between $R \oplus R$ and $M \oplus M$ [even though $M \not\cong R$].

(c) Show that $f \mapsto fs \otimes s + fc \otimes c$ is an isomorphism between R and $M \otimes_R M$ [even though $M \not\cong R$]. [**Hint:** Find an inverse.]

2. Let R be a commutative with $1 \neq 0$ and M an R-module. Show that $\mathrm{Hom}_R(R \oplus R, M)$ is projective if, and only if, M is a projective R-module.

3. Let $f(x) \in F[x]$ be irreducible, with $F \subseteq \mathbb{R}$. Suppose that there is $\alpha_0 \in \mathbb{C} - \mathbb{R}$ such that $f(\alpha_0) = 0$ and $|\alpha_0| = 1$. Show that if α is a root of $f(x)$, then so is $1/\alpha$.

4. Let ζ_n be a primitive n-th root of unity, and $\alpha \in \mathbb{Q}[\zeta_n] \cap \mathbb{R}$, such that $\alpha^m \in \mathbb{Q}$ for some $m \geq 2$. Show that $\alpha^2 \in \mathbb{Q}$.