1) Compute the following limits:

(a) [5 points] $\lim_{x\to 0} \frac{\sin(x^2)}{\ln(x+1)}$

Solution.

$$\lim_{x\to 0} \frac{\sin(x^2)}{\ln(x+1)} = \lim_{x\to 0} \frac{\cos(x^2)\cdot 2x}{1/(x+1)}$$

$$= 0.$$

□

(b) [5 points] $\lim_{x\to 0} x\ln(x^2)$

Solution.

$$\lim_{x\to 0} x\ln(x^2) = \lim_{x\to 0} \frac{\ln(x^2)}{1/x}$$

$$= \lim_{x\to 0} \frac{1/x^2 \cdot 2x}{-1/x^2}$$

$$= \lim_{x\to 0} -2x = 0.$$

□

(c) [5 points] $\lim_{x\to 1} \frac{x^2+1}{(x-1)^2}$

Solution. Trying to plug in $x = 1$ we get "2/0", so the limit is some kind of infinite limit. Since the numerator and denominator are always non-negative, the limit is $+\infty$. □

2) Compute the derivatives of the following functions. [**No need to simplify your answers!**]

(a) [5 points] $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{\sqrt{x}}{\cos(x)}\right)$

Solution.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\sqrt{x}}{\cos(x)}\right) = \frac{\frac{1}{2\sqrt{x}} \cdot \cos(x) - \sqrt{x} \cdot (-\sin(x))}{(\cos(x))^2}$$

$\square$

(b) [5 points] $\dfrac{\mathrm{d}}{\mathrm{d}x}\big(\ln(x^2+1) \cdot \arctan(x)\big)$

Solution.

$$\frac{\mathrm{d}}{\mathrm{d}x}\big(\ln(x^2+1) \cdot \arctan(x)\big) = \frac{1}{x^2+1} \cdot 2x \cdot \arctan(x) + \ln(x^2+1) \cdot \frac{1}{x^2+1}.$$

$\square$

(c) [5 points] $\dfrac{\mathrm{d}}{\mathrm{d}x}\big((\sin(2^x))^5\big)$

Solution.

$$\frac{\mathrm{d}}{\mathrm{d}x}\big((\sin(2^x))^5\big) = 5\,(\sin(2^x))^4 \cdot \cos(2^x) \cdot 2^x \cdot \ln(2).$$

$\square$

3) Consider the curve given by the equation:

$$x^2 - y^2 = 1$$

(a) [5 points] Find the equation of the tangent line at the point $(2, \sqrt{3})$.

Solution. We have:

$$2x - 2yy' = 0, \qquad \text{and so} \qquad y' = \frac{x}{y}.$$

So, the equation is:

$$y - \sqrt{3} = \frac{2}{\sqrt{3}}(x - 2).$$

$\square$

(b) [5 points] Is the curve concave up or concave down at $(2, \sqrt{3})$? [**Justify or you get zero!**]

Solution. We have:

$$y'' = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x}{y}\right) = \frac{1 \cdot y - x \cdot y'}{y^2} = \frac{y - x^2/y}{y^2} = \frac{y^2 - x^2}{y^3}.$$

At $(2, \sqrt{3})$ we get $-\frac{1}{3\sqrt{3}} < 0$, so it is concave down. $\square$

4) [5 points] Find the equation of the line tangent to the curve given by

$$x(t) = t^2 - t$$

$$y(t) = \sqrt{t} + \frac{1}{t}$$

at $t = 1$.

Solution. We have:

$$x'(t) = 2t - 1$$

$$y'(t) = \frac{1}{2\sqrt{t}} - \frac{1}{t^2}$$

So, at $t = 1$ the slope is $y'(1)/x'(1) = (-1/2)/1 = -1/2$. Since $x(1) = 0$ and $y(1) = 2$, the equation of the tangent line is:

$$y - 2 = -\frac{1}{2}(x - 0).$$

$\square$

5) Let $f(x) = x^3 - 4x$.

(a) [2 points] Find the x and y-intercepts of $f(x)$.

Solution. y-intercept is $f(0) = 0$.

Since $f(x) = x^3 - 4x = x(x-2)(x+2) = 0$ gives $x = -2, 0, 2$, these are the x-intercept. $\square$

(b) [5 points] Give the intervals where $f(x)$ is increasing and decreasing and its critical points [if any], classifying them as local maximum, local minimum or neither.

Solution. We have $f'(x) = 3x^2 - 4$, so the critical points are $\pm 2/\sqrt{3}$. Analyzing the signs we obtain positive [so $f(x)$ is increasing] on $(-\infty, -2/\sqrt{3})$ and $(2/\sqrt{3}, \infty)$, and negative [and so $f(x)$ is decreasing] on $(-2/\sqrt{3}, 2/\sqrt{3})$. Thus $-2\sqrt{3}$ is a local maximum and $2/\sqrt{3}$ is a local minimum. $\square$

(c) [5 points] Give the intervals where $f(x)$ is concave up and concave down and its inflection points [if any].

Solution. We have that $f''(x) = 6x$. So it is positive [and hence $f(x)$ is concave up] in $(0, \infty)$ and it is negative [and hence $f(x)$ is concave down] in $(-\infty, 0)$. Thus, 0 is an inflection point. $\square$

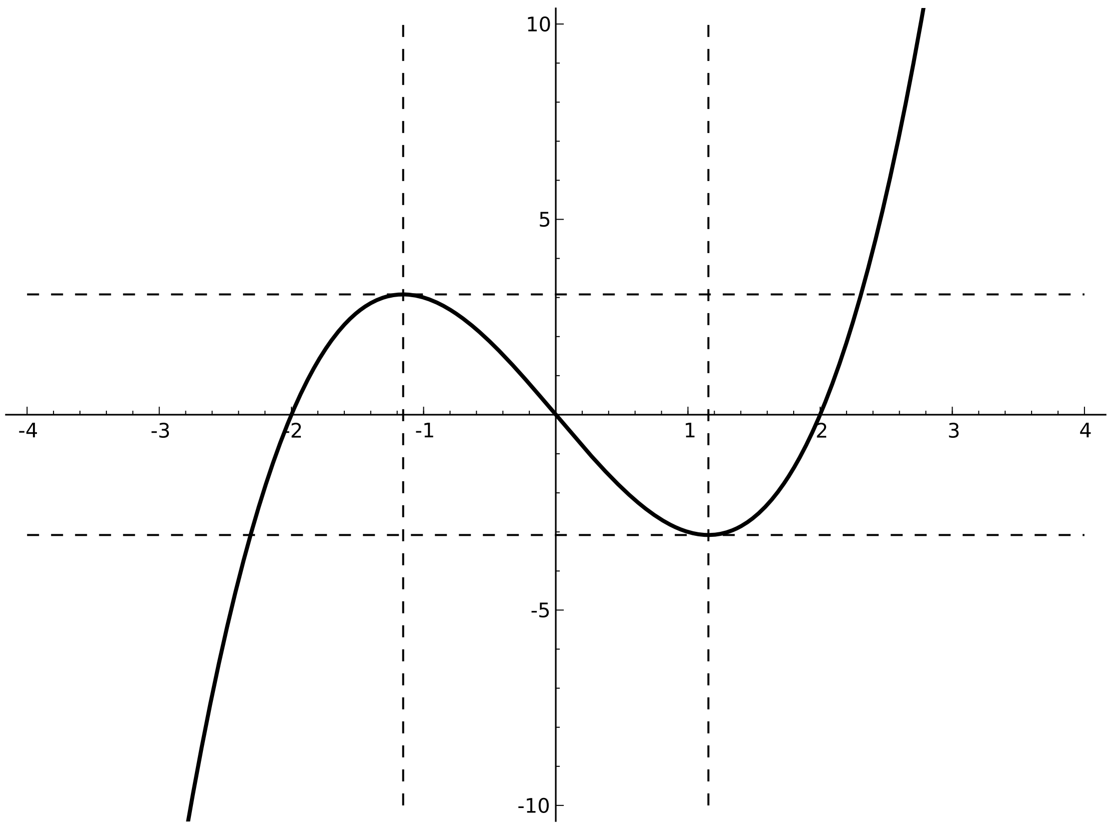
(d) [5 points] Compute $\lim\limits_{x\to\infty} f(x)$ and $\lim\limits_{x\to-\infty} f(x)$

Solution. We have:

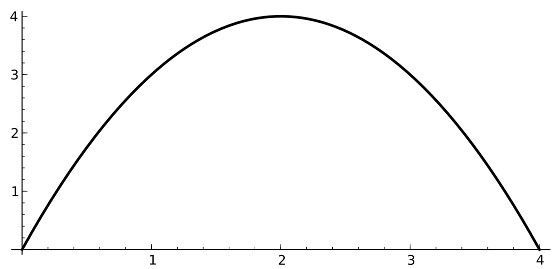
$$\lim_{x\to-\infty} x^3 - 4x = -\infty \qquad \text{and} \qquad \lim_{x\to\infty} x^3 - 4x = \infty.$$

$\square$

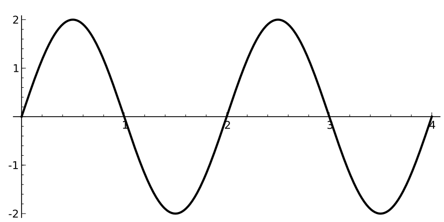
(e) [8 points] Sketch the graph of $f(x)$ below. [**Hint:** $2/\sqrt{3} = 2\sqrt{3}/3 \approx 1.547$ and $f(2/\sqrt{3}) \approx -3.079$.]

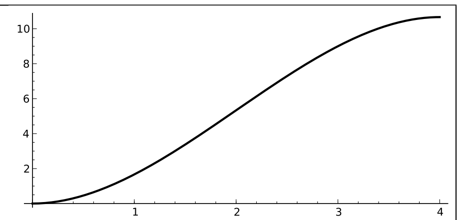


6) [10 points] Let $v(t)$ be the velocity of a particle moving along a straight line at time t. The picture below is the graph of $v(t)$:

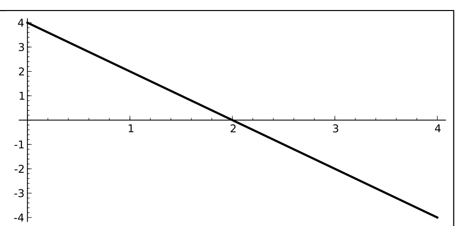


Which of the graphs below could represent the position of the particle and which could represent the acceleration? [Label the graph clearly!]

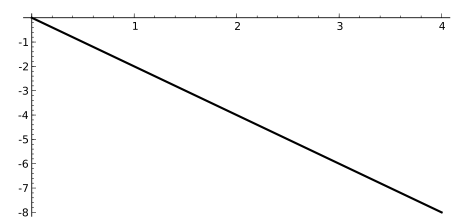


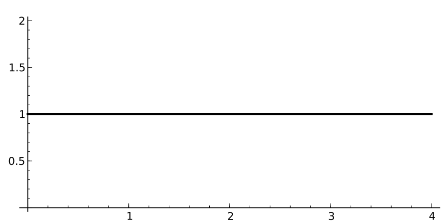


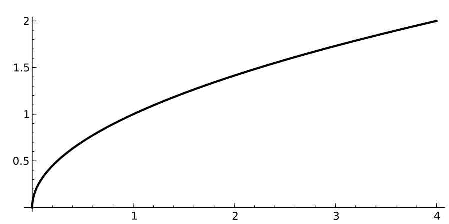
posit.



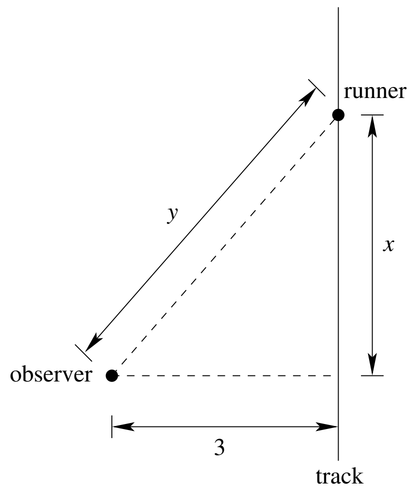
accel.







7) [10 points] A person in the audience watches a runner pass by in a long straight stretch. [See the picture below.] The person is 3 meters away from the track and the runner is running at a constant speed of 2.5 meters per second. [The runner is going *up* in the picture below, i.e., x is increasing.] How fast is the distance between the runner and the person in the audience [labeled y in the picture] increasing when the runner is 5 meters away [i.e., when $y = 5$]?



Solution. We have $3^2 + x^2 = y^2$. Taking derivatives we get $2xx' = 2yy'$, which implies $y' = xx'/y = 2.5 \cdot x/y$.

When $y = 5$ we get $x = 4$ [from the equation above] and so, at that moment we get $y' = 2.5 \cdot 4/5 = 2$ meters per second. □

8) [10 points] You need to build a box with a *square base*, with both bottom and top [lid]. The box must have volume 16 cubic inches. The cost of the material for the sides of the box is 1 dollar per square inch, and the cost for the sturdier material for the top and bottom of the box is 2 dollars per square inch. What is the minimal cost to build this box?

Solution. Let x be the length of the side of the square bottom [and top] and y be the height of the box. Then, the total cost is:

$$C = 2 \cdot x^2 \cdot 2 + 4 \cdot xy \cdot 1 = 4x^2 + 4xy.$$

Now, the volume gives us:

$$16 = x^2 y, \qquad \text{and so} \qquad y = \frac{16}{x^2}.$$

So, we have that cost, in terms of x is:

$$C = 4x^2 + \frac{64}{x}, \qquad \text{with } x \in (0, \infty).$$

So, $C' = 8x - 64/x^2 = 0$. Simplifying, it gives us $x^3 = 8$, i.e., $x = 2$. Analyzing the sign of the derivative we see that $x = 2$ is a global minimum.

So, the minimal cost is 48 dollars. □